

Практика 4

Луа Yaroshevskiy

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- $x = ar \cos^\alpha \varphi$
- $y = br \sin^\alpha \varphi$

$$J = \alpha ab r \cos^{\alpha-1} \varphi \sin^{\alpha-1} \varphi \quad (1)$$
$$\frac{y}{x} = \text{const} \cdot \text{tg}^\alpha \varphi$$

Задача 1.

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} = \frac{x^2}{h^2} + \frac{y^2}{k^2} \quad x = 0, y = 0$$

Решение.

$$\alpha = \frac{2}{3} \quad r = \frac{a^2}{h^2} \cos^{\frac{4}{3}} \varphi + \frac{b^2}{k^2} \sin^{\frac{4}{3}} \varphi$$

Из 1 $\alpha = \frac{2}{3}$

$$\begin{aligned} \lambda\Omega &= \iint_{\Omega} 1 dx dy = \int_0^{\frac{\pi}{2}} du \int_0^{r(u)} \frac{2}{3} ab r \cos^{\alpha-1} \varphi \sin^{\alpha-1} \varphi dr = \\ &= \frac{1}{3} ab \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} \varphi \sin^{\alpha-1} \varphi \left(\frac{a^2}{h^2} \cos^{\frac{4}{3}} \varphi + \frac{b^2}{k^2} \sin^{\frac{4}{3}} \varphi \right)^2 d\varphi = \text{I} + \text{II} + \text{III} \\ \text{I} &= \int_0^{\frac{\pi}{2}} \frac{a^4}{h^4} \cos^{\frac{7}{3}} \varphi \sin^{-\frac{1}{3}} \varphi d\varphi = \left| d\varphi = \frac{t = \sin^2 \varphi}{2t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} = \frac{a^4}{b^4} \int_0^1 (1-t)^{\frac{7}{6}-\frac{1}{2}} t^{-\frac{1}{6}-\frac{1}{2}} = \frac{a^4}{2b^4} B\left(\frac{5}{3}, \frac{1}{3}\right) \right. \end{aligned}$$

Задача 2.

$$(x^3 + y^3)^2 = x^2 + y^2$$

, $x \geq 0, y \geq 0 \rightarrow$ полярные координаты

Решение.

$$\begin{aligned} r^4(\cos^3 \varphi + \sin^3 \varphi) &= 1 \\ \lambda(\Omega) &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{r(\varphi)} \underset{\text{Якобиан}}{r} dr = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\cos^3 \varphi + \sin^3 \varphi} = \frac{1}{2} \int_0^{+\infty} \frac{\sqrt{t^2+1}}{t^3+1} dt \\ &= \int \frac{d\varphi}{(\cos \varphi + \sin \varphi)(1 - \cos \varphi \sin \varphi)} \\ &= \int \frac{dt}{\left(\frac{2t^2}{1+t^2}\right)^3 + \left(\frac{2t}{1+t^2}\right)^3} \cdot \frac{2}{1+t^2} \end{aligned}$$

Задача 3.

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$$

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 4$$

$$, \frac{x}{a} = \frac{y}{b}, 8\frac{x}{a} = \frac{y}{b}$$

Решение.

$$x = ar \cos^3 \varphi$$

$$y = br \sin^3 \varphi$$

- $r = 1, r = 8$
- $\varphi = 1, \varphi = \operatorname{arctg} 2$

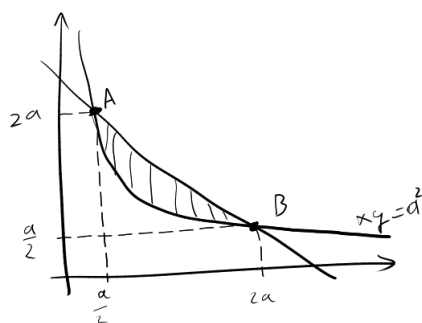
$$\lambda(\Omega) = \int_1^8 dr \int_1^{\operatorname{arctg} 2} 3abr \cos^2 \varphi \sin^2 \varphi d\varphi = \frac{3}{2}abr^2 \int_1^{\operatorname{arctg} 2} \underbrace{\sin^2 2\varphi}_{\frac{1 - \cos 4\varphi}{2}} d\varphi$$

Задача 4.

$$xy = a^2$$

$$x + y = \frac{5}{2}$$

Решение.



$$\int_{\frac{a}{2}}^{2a} \int_{\frac{a^2}{x}}^{\frac{5}{2}a-x} dy = \int_{\frac{a}{2}}^{2a} \left(\frac{5}{2}a - x - \frac{a^2}{x} \right) dx = \left. \frac{5}{2}ax - \frac{x^2}{2} - a^2 \ln x \right|_{\frac{a}{2}}^{2a}$$

Задача 5.

$$xy = a^2$$

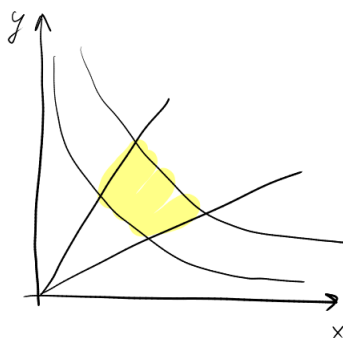
$$xy = 2a^2$$

$$y = x$$

$$y = 2x$$

$$, x, y > 0$$

Решение.



$$u = xy, v = \frac{y}{x}$$

$$x = \sqrt{\frac{u}{v}}$$

$$y = \sqrt{uv}$$

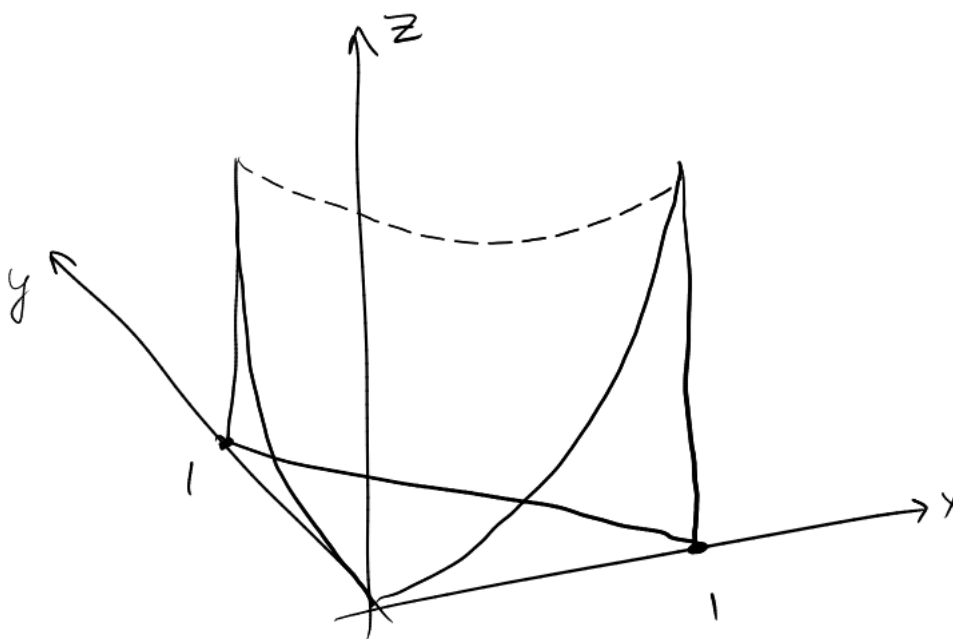
$$J = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}} \\ \frac{1}{2}\frac{\sqrt{v}}{\sqrt{u}} & \frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$$

$$\lambda(\Omega) = \int_{a^2}^{2a^2} du \int_1^2 \frac{1}{2v} dv = a^2 \int_1^2 \frac{1}{2v} dv = \frac{a^2}{2} \ln 2$$

Задача 6.

$$V = \int_0^1 dx \int_0^{1-x} x^2 + y^2 dy$$

Определим что эта за фигура в плоскости xOy

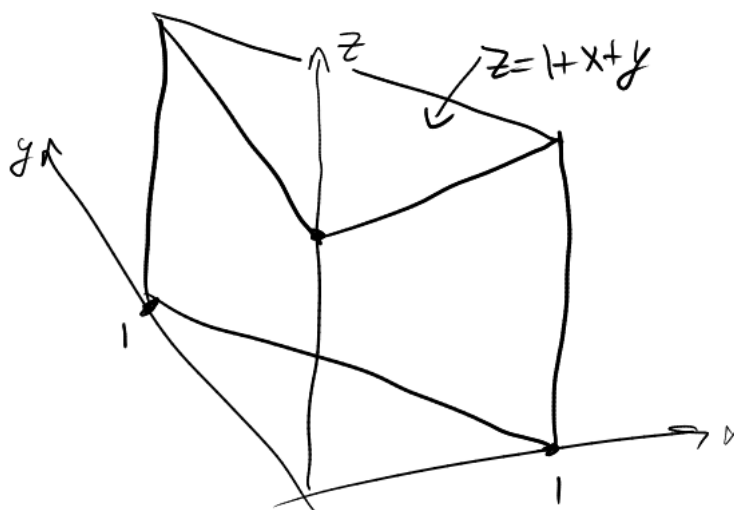


Задача 7.

$$\begin{cases} z = 1 + x + y \\ z = 0 \end{cases}$$

$$\begin{cases} x + y = 1 \\ x = 0 \\ y = 0 \end{cases}$$

Цилиндрическая форма

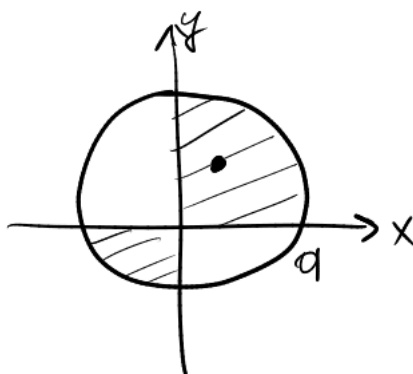


Решение.

$$\lambda(\Omega) = \int_0^1 dx \int_0^{1-x} 1 + x + y dy = \int_0^1 (1+x)(1-x) + \frac{y^2}{2} \Big|_0^{1-x} dx$$

Задача 8.

$$\begin{cases} z^2 = xy \\ x^2 + y^2 = a^2 \end{cases}$$



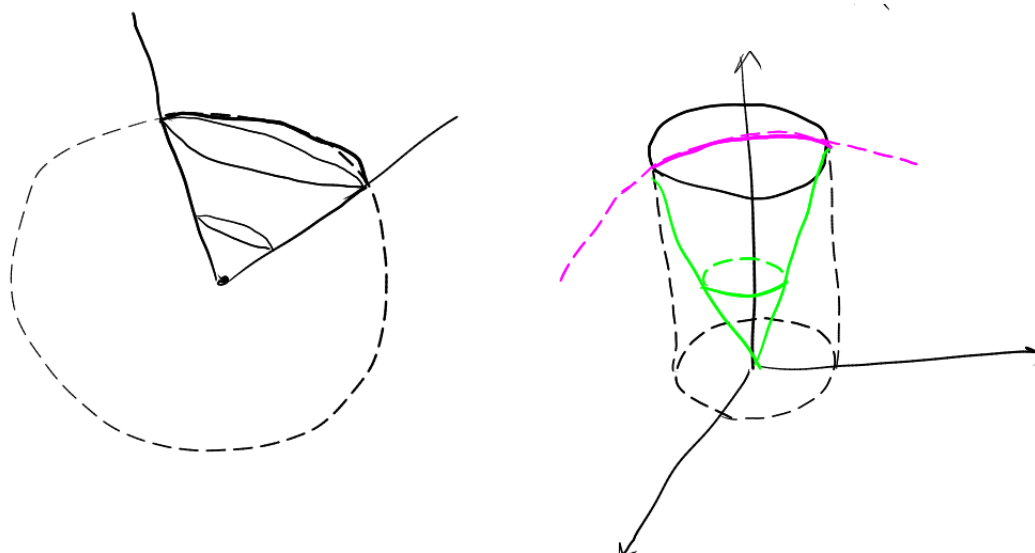
Решение.

$$\lambda(\Omega) = 4 \iint \sqrt{xy} dx dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r^2 \sqrt{\cos \varphi \sin \varphi} dr = \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}} \varphi \sin^{\frac{1}{2}} \varphi d\varphi$$

Задача 9.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$



Решение.

$$V = \iint z_2(x, y) - z_1(x, y) \quad (2)$$

$$\frac{2z^2}{c^2} = 1$$

$$z = \frac{c}{\sqrt{2}}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$$

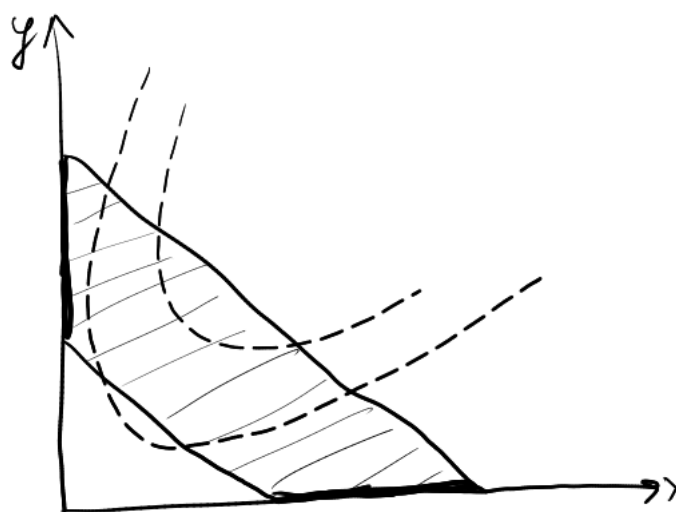
$$2 = \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} ab r (-c^2 \cdot r^2 + c^2(1 - r^2)) = 2\pi \int_0^{\frac{1}{\sqrt{2}}}$$

Задача 10.

$$z^2 = xy$$

$$x + y = a$$

$$x + y = b$$



Решение.

$$2 \int_a^b dx \int_0^{a-x} \sqrt{xy} dy + \int_0^a dx \int_{a-x}^{b-x} \sqrt{xy} dy$$